Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of
APPLIED
MATHEMATICS
AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 253–262

# FEATURES OF THE DYNAMIC STRESSES IN THE NEIGHBOURHOOD OF THE JOINT OF THREE ELASTIC MEDIA†

L. P. VOVK and B. V. SOBOL'

Rostov-on-Don

e-mail: bsobol@dstu.edu.ru; vovklp@ukrtop.com

(*Received* 21 *November* 2004)

The solution of the problem of the harmonic oscillations of a piecewise-inhomogeneous domain, consisting of three joined rectangles with different elastic properties, is constructed within the framework of a modified superposition method. The discontinuities in the wave field are investigated in the neighbourhood of a singular point of the boundary at the joint of the rectangles. 

A knowledge of the nature of the behaviour of the components of the stress–strain state close to singular points and lines of the surface of the body being considered enables one to approximate the solution of problems in the theory of elasticity in an optimal manner and to construct an efficient numerical algorithm in order to find it. This undertaking is even more urgent in problems of the vibration loading of structural components when the stressed state can undergo qualitative changes depending on the frequency of the external load. The discontinuities in the distribution of the static stresses in the neighbourhood of the corner point of the line of separation of the domains of the cross-section of a body, composed of two different prismatic bodies which have been joined along a lateral surface, have been considered earlier in [1–3]. For example, the plane, elastostatic problem of two unlike wedges with arbitrary aperture angles was considered in [1]; the solution was constructed in terms of Mellin transforms which, after satisfying the matching conditions, enables one to investigate the dependence of the order of the singularity of the stress field at the vertex of the wedges on the aperture angles and combinations of the constants of elasticity. A method was described in [4] which enables one to establish the nature of the above-mentioned discontinuities without solving the boundary-value problem directly. Dynamic aspects of the problem have been considered in [5–7] and, in particular, the concept of a "boundary" resonance, which is a generalization of the thoroughly investigated edge resonance [8], was introduced.

The problem of determining the qualitative and quantitative nature of the discontinuity in the wave field, which arises in the neighbourhood of the corner point of the joint of three unlike regions of rectangular form, is presented below. Such problems arise when calculating the strength parameters of welded or soldered butt joints with angular joints [9]. The general solution of the problem of the harmonic oscillations of an inhomogeneous rectangle with an internal aperture was constructed earlier in [10] using a modification of the superposition method, which uses the asymptotic behaviour of the wave characteristics at singular points of the boundary.

†*Prikl. Mat. Mekh.* Vol. 69, No. 2, pp. 279–289, 2005.

0021–8928/$—see front matter. 
doi: 10.1016/j.jappmathmech.2005.03.011

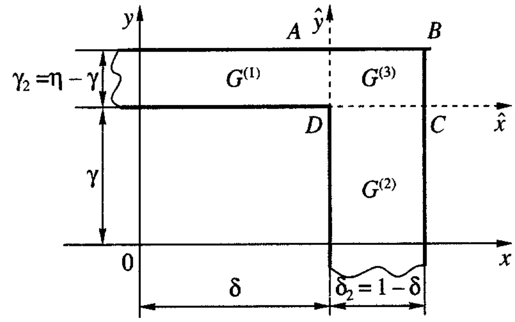


Fig. 1

## 1. FORMULATION OF THE PROBLEM

Suppose a section of a piecewise-inhomogeneous elastic prism, which is infinite in the direction of the $\alpha_3$ axis, occupies a region $\alpha_1 O \alpha_2$ in the system of coordinates $D = \bar{G}^{(1)} \cup \bar{G}^{(2)} \cup \bar{G}^{(3)}$, where the regions $\bar{G}^{(m)}$ are joined to one another and are defined by the inequalities

$$\bar{G}^{(1)} = \{(\alpha_1, \alpha_2) : |\alpha_1| \le c, \ \alpha_2 \in [-b, -d] \cup [d, b]\}$$

$$\bar{G}^{(2)} = \{(\alpha_1, \alpha_2) : \alpha_1 \in [-a, -c] \cup [c, a], \ |\alpha_2| \le d\}$$

$$\bar{G}^{(3)} = \{(\alpha_1, \alpha_2) : \alpha_1 \in [-a, -c] \cup [c, a], \ \alpha_2 \in [-b, -d] \cup [d, b]\}$$

The material of the regions $\bar{G}^{(m)}$ is assumed to be isotropic and is defined by their shear modulus $\mu^{(m)}$, Poisson's ratio $\nu^{(m)}$ and the density $\rho^{(m)}$. Henceforth, a superscript will mean that a mechanical characteristic or an elastic modulus belongs to the region $\bar{G}^{(m)}$ ($m = 1, 2, 3$), and Greek indices take the values of 1 and 2.

Suppose a vibrating load of varying intensity $q$, which varies harmonically with time with a frequency $\omega$, is specified on the external sides of the section $\alpha_1 = \pm a$, $\alpha_2 = \pm b$ and that the internal boundary of the section is free.

For convenience, we will introduce the local dimensionless coordinates

$$\hat{x} = (\alpha_1 - c)/a, \quad \hat{y} = (\alpha_2 - d)/a$$

in the region of the section and the dimensionless geometric parameters

$$\eta = b/a, \quad \delta = c/a, \quad \gamma = d/a, \quad \delta_2 = 1 - \delta, \quad \gamma_2 = \eta - \gamma$$

By taking account of the symmetry of the region $D$, the wave field of the part of the region which is located in the first quadrant can be considered. This part of the region is shown in Fig. 1 in dimensionless coordinates.

The dimensionless amplitude components of the stress tensor $\sigma_{\alpha\beta}^{(m)}$, referred to $\mu^{(m)}$, are related to the dimensionless displacements $U_\beta^{(m)}$ referred to $a$, by Hooke's law for an isotropic body and depend on the dimensionless frequency parameter $\Omega^{(m)} = \omega a / \sqrt{\mu^{(m)}/\rho^{(m)}}$.

The boundary conditions of the problem include power conditions for the load on the external boundary of the section and the condition for fast coupling of the regions $G^{(m)}$. They can be written in dimensionless form as follows:

in the domain $G^{(1)} = \{|x| \le \delta; \ 0 \le \hat{y} \le \gamma_2\}$

$$\sigma_{1\beta}^{(1)}(\delta, \hat{y}) = r_{31} \sigma_{1\beta}^{(3)}(0, \hat{y}), \quad U_\beta^{(1)}(\delta, \hat{y}) = U_\beta^{(3)}(0, \hat{y})$$

$$\sigma_{22}^{(1)}(x, \gamma_2) = q^{(1)}, \quad \sigma_{12}^{(1)}(x, \gamma_2) = \sigma_{12}^{(1)}(x, 0) = \sigma_{22}^{(1)}(x, 0) = 0$$

in the domain $G^{(2)} = \{0 \le \hat{x} \le \delta_2;\ |y| \le \gamma\}$

$$\begin{aligned} &\sigma_{\beta 2}^{(2)}(\hat{x}, \gamma) = r_{32}\sigma_{\beta 2}^{(3)}(\hat{x}, 0), \quad U_\beta^{(2)}(\hat{x}, \gamma) = U_\beta^{(3)}(\hat{x}, 0) \\ &\sigma_{11}^{(2)}(\delta_2, y) = q^{(2)}, \ \sigma_{12}^{(2)}(\delta_2, y) = \sigma_{12}^{(2)}(0, y) = \sigma_{11}^{(2)}(0, y) \end{aligned} \tag{1.1}$$

in the domain $G^{(3)} = \{0 \le \hat{x} \le \delta_2;\ 0 \le \hat{y} \le \gamma_2\}$

$$\sigma_{11}^{(3)}(\delta_2, \hat{y}) = q^{(3)}, \quad \sigma_{12}^{(3)}(\delta_2, \hat{y}) = 0, \quad \sigma_{22}^{(3)}(\hat{x}, \gamma_2) = q^{(3)}, \quad \sigma_{12}^{(3)}(\hat{x}, \gamma_2) = 0$$

Here,

$$r_{ij} = \mu^{(i)}/\mu^{(j)}, \quad q^{(m)} = q/\mu^{(m)}$$

## 2. CONSTRUCTION OF THE GENERAL SOLUTION

We will construct the general solution $U_\beta^{(m)}$, which satisfies the system of equations of motion within the regions $G^{(m)}$, using the superposition method [8] in the form of the sum of two partial solutions of this system, each of which describes the oscillations of infinite strips which form the domain $G^{(m)}$ at their intersection. The evenness or oddness of these partial solutions is determined by the form of the boundary conditions. At the same time, it is necessary to take account of the fact that the functions $U_\beta^{(1)}(x, \hat{y})$ and $U_\beta^{(3)}(\hat{x}, \hat{y})$ with respect to the coordinate $\hat{y}$ and the functions $U_\beta^{(2)}(\hat{x}, y)$ and $U_\beta^{(3)}(\hat{x}, \hat{y})$ with respect to the coordinate $\hat{x}$ are functions of a common form. Hence, the general solution of the problem in the regions $G^{(m)}$ is written in the form

$$\begin{aligned} U_1^{(1)} &= H_1^{(1)}\operatorname{sh}(t^{(1)}x)\cos\theta^{(1)}(\hat{y} - \gamma_2) + \tilde{U}_1^{(1)}\sin\chi^{(1)}(x - \delta) \\ U_2^{(1)} &= H_2^{(1)}\operatorname{ch}(t^{(1)}x)\sin\theta^{(1)}(\hat{y} - \gamma_2) + \tilde{U}_2^{(1)}\cos\chi^{(1)}(x - \delta) \\ U_1^{(2)} &= \hat{U}_1^{(2)}\cos\theta^{(2)}(y - \gamma) + R_1^{(2)}\operatorname{ch}(l^{(2)}y)\sin\chi^{(2)}(\hat{x} - \delta_2) \\ U_2^{(2)} &= \hat{U}_2^{(2)}\sin\theta^{(2)}(y - \gamma) + R_2^{(2)}\operatorname{sh}(l^{(2)}y)\cos\chi^{(2)}(\hat{x} - \delta_2) \\ U_1^{(3)} &= \hat{U}_1^{(3)}\cos\theta^{(1)}(\hat{y} - \gamma_2) + \tilde{U}_1^{(3)}\sin\chi^{(2)}(\hat{x} - \delta_2) \\ U_2^{(3)} &= \hat{U}_2^{(3)}\sin\theta^{(1)}(\hat{y} - \gamma_2) + \tilde{U}_2^{(3)}\cos\chi^{(2)}(\hat{x} - \delta_2) \end{aligned} \tag{2.1}$$

where

$$\tilde{U}_\beta^{(n)} = R_\beta^{(n)}\operatorname{sh}(l^{(n)}\hat{y}) + S_\beta^{(n)}\operatorname{ch}(l^{(n)}, \hat{y}), \ n = 1, 3; \ \hat{U}_\beta^{(p)} = H_\beta^{(p)}\operatorname{sh}(t^{(p)}\hat{x}) + Q_\beta^{(p)}\operatorname{ch}(t^{(p)}, \hat{x}), \ p = 2, 3$$

The set of constants $H_\beta^{(m)}$, $Q_\beta^{(m)}$, $R_\beta^{(m)}$ and $S_\beta^{(m)}$ in formulae (2.1) ensures the necessary degree of arbitrariness for satisfying the boundary conditions and the matching conditions (1.1) in the composite domain being considered. It is advisable to choose sequences of numbers $\theta_k^{(\beta)}$ and $\chi_j^{(\beta)}$ as the values of $\theta^{(\beta)}$ and $\chi^{(\beta)}$ such that the systems of corresponding functions are complete and orthogonal in the corresponding intervals [8, 10]. From this requirement, the values

$$\theta_k^{(1)} = k\pi/\gamma_2, \ \theta_k^{(2)} = k\pi/\gamma, \ \chi_j^{(1)} = j\pi/\delta, \ \chi_j^{(2)} = j\pi/\delta_2; \ k = 1, 2, \ldots; \ j = 1, 2, \ldots$$

follow as being possible.

Substituting expression (2.1) into the systems of equations of motion, we obtain systems of linear homogeneous equations in the coefficients $H_1^{(m)}$ and $H_2^{(m)}, \ldots, S_1^{(m)}$ and $S_2^{(m)}$ for each value of $k$ and $j$. From the condition for a non-trivial solution of these systems to exist, we find the values of the parameters $t^{(m)}$ and $l^{(m)}$.

$$t_{\beta k}^{(m)^2} = \theta_k^{(m)^2} - \Omega_\beta^{(m)^2}, \quad l_{\beta j}^{(m)^2} = \chi_j^{(m)^2} - \Omega_\beta^{(m)^2}$$

where

$$\Omega_1^{(m)^2} = \Omega^{(m)^2}/N_{11}^{(m)}, \quad N_{11}^{(m)} = 2(1-\nu^{(m)})/(1-2\nu^{(m)}), \quad \Omega_2^{(m)} = \Omega^{(m)}$$
$$\theta_k^{(1)} = \theta_k^{(3)}, \quad \chi_j^{(2)} = \chi_j^{(3)}$$

and the relation between the above-mentioned coefficients, which completely defines the general solution of the problem in all of the regions $G^{(m)}$ and enables one to satisfy the matching conditions and the power boundary conditions.

## 3. THE SOLUTION OF SUBSIDIARY PROBLEMS

In accordance with the algorithm for the modified superposition method, which was first proposed for the case of homogeneous, finite regions [11] and for a domain extended into non-homogeneous domains [7, 10], we replace the initial boundary conditions with subsidiary conditions. This enables us to obtain an analytical solution of the subsidiary problem. The solution of the initial boundary-value problem will be expressed in terms of additional functions, which specify the boundary conditions which have been introduced. The rules for the change in these functions in the neighbourhood of the singular points of the domain enable us to investigate the singularities in the concentration of the stresses and to separate out the slowly converging parts in the series for all of the wave characteristics. In the case being considered the boundary conditions of the subsidiary problem is complicated considerably in view of the two internal lines of separation of the domains $G^{(m)}$ and take the following form

$$\begin{aligned}
&G^{(1)} = \{|x| \le \delta;\ 0 \le \hat{y} \le \gamma_2\}:\\
&U_1^{(1)}(\delta, \hat{y}) = f_1(\hat{y}), \quad \sigma_{12}^{(1)}(\delta, \hat{y}) = \varphi_1(\hat{y})\\
&U_2^{(1)}(x, \gamma_2) = f_2(x), \quad \sigma_{12}^{(1)}(x, \gamma_2) = 0, \quad U_2^{(1)}(x, 0) = f_3(x), \quad \sigma_{12}^{(1)}(x, 0) = 0\\
&G^{(2)} = \{0 \le \hat{x} \ge \delta_2;\ |y| \le \gamma\}:\\
&U_1^{(2)}(\delta_2, y) = f_4(y), \quad \sigma_{12}^{(2)}(\delta_2, y) = 0, \quad U_1^{(2)}(0, y) = f_5(y), \quad \sigma_{12}^{(2)}(0, y) = 0 \qquad (3.1)\\
&U_2^{(2)}(\hat{x}, \gamma) = f_6(\hat{x}), \quad \sigma_{12}^{(2)}(\hat{x}, \gamma) = \varphi_2(\hat{x})\\
&G^{(3)} = \{0 \le \hat{x} \le \delta_2;\ 0 \le \hat{y} \le \gamma_2\}:\\
&U_1^{(3)}(\delta_2, \hat{y}) = f_7(\hat{y}), \quad \sigma_{12}^{(3)}(\delta_2, \hat{y}) = 0, \quad U_1^{(3)}(0, \hat{y}) = f_1(\hat{y}), \quad \sigma_{12}^{(3)}(0, \hat{y}) = r_{13}\varphi_1(\hat{y})\\
&U_2^{(3)}(\hat{x}, \gamma_2) = f_8(\hat{x}), \quad \sigma_{12}^{(3)}(\hat{x}, \gamma_2) = 0, \quad U_2^{(3)}(\hat{x}, 0) = f_6(\hat{x}), \quad \sigma_{12}^{(3)}(\hat{x}, 0) = r_{23}\varphi_2(\hat{x})
\end{aligned}$$

The unknown subsidiary functions are denoted by $f_1(\hat{y}), \varphi_1(\hat{y}), \ldots, f_8(\hat{y})$.

Note that the choice of the boundary conditions of the subsidiary problem in the form (3.1) enables us automatically to satisfy the parts of the boundary conditions of the initial boundary-value problem which touch upon the normal displacements and shear stresses in the external and internal boundaries of the domain. We expand the subsidiary functions in Fourier series in the corresponding intervals and, using the general solution of the problem, we set up conditions (3.1). The resulting sets of linear systems admit of an analytical solution and enable us to express, in explicit form, the characteristics of the wave field in the whole of the composite region of the section in terms of the Fourier coefficients $f_{10}, f_{1k}, f_{20}, f_{2j}, \varphi_{1k}, \ldots$ of the subsidiary functions which have been introduced. For example, the expressions for the displacements in the region $G^{(1)}$ have the form (summation with respect to $k$ and $j$ is carried out from one to infinity everywhere)

$$U_1^{(1)} = \frac{1}{\Omega_2^{(1)2}}\left\{\sum_k V_{k65}^{(1)}\cos\theta_k^{(1)}(\hat{y}-\gamma_2) + \sum_k W_{j3}^{(1)}\sin\chi_j^{(1)}(x-\delta)\right\} + f_{10}\frac{\sin\Omega_1^{(1)}x}{\sin\Omega_1^{(1)}\delta}$$

$$U_2^{(1)} = \frac{1}{\Omega_2^{(1)2}}\left\{\sum_k V_{k34}^{(1)}\sin\theta_k^{(1)}(\hat{y}-\gamma_2) + \sum_k W_{j6}^{(1)}\cos\chi_j^{(1)}(x-\delta)\right\} +$$

$$+ f_{20}\frac{\sin\Omega_1^{(1)}\hat{y}}{\sin\Omega_1^{(1)}\gamma_2} - f_{30}\frac{\sin\Omega_1^{(1)}(\hat{y}-\gamma_2)}{\sin\Omega_1^{(1)}\gamma_2}$$

where

$$V_{krs}^{(1)} = 2(\theta_k^{(1)})^2 f_{1k}\Delta_r^{(1)}(x, \delta, \theta_k^{(1)}) + \theta_k^{(1)}\varphi_{1k}\Delta_s^{(1)}(x, \delta, \theta_k^{(1)}), \quad rs = 65, 34$$

$$W_{jt}^{(1)} = 2(\chi_j^{(1)})^2[f_{2j}\Delta_t^{(1)}(\hat{y}, \gamma_2, \chi_j^{(1)}) - f_{3j}\Delta_t^{(1)}(\hat{y}-\gamma_2, \gamma_2, \chi_j^{(1)})], \quad y = 3, 6$$

$$\Delta_3^{(m)}(u, v, z_j) = \frac{a_{3j}^{(m)2}}{2z_j a_{1j}^{(m)}}C_{1j}^{(m)} - \frac{a_{2j}^{(m)}}{z_j}C_{2j}^{(m)}, \quad \Delta_4^{(m)}(u, v, z_j) = \frac{z_j}{a_{1j}^{(m)}}C_{1j}^{(m)} - \frac{a_{2j}^{(m)}}{z_j}C_{2j}^{(m)}$$

$$\Delta_5^{(m)}(u, v, z_j) = S_{2j}^{(m)} - S_{1j}^{(m)}, \quad \Delta_6^{(m)}(u, v, z_j) = S_{2j}^{(m)} - \frac{a_{3j}^{(m)2}}{2z_j^2}S_{1j}^{(m)}$$

$$S_{\gamma j}^{(m)} = \frac{\operatorname{sh} a_{\gamma j}^{(m)}u}{\operatorname{sh} a_{\gamma j}^{(m)}v}, \quad C_{\gamma j}^{(m)} = \frac{\operatorname{ch} a_{\gamma j}^{(m)}u}{\operatorname{sh} a_{\gamma j}^{(m)}v}, \quad a_{\beta j}^{(m)2} = z_j^2 - \Omega_\beta^{(m)2}, \quad a_{3j}^{(m)2} = a_{2j}^{(m)2} + z_j^2$$

The form of writing the solution of the subsidiary problems which has been described assumes that all those frequency values for which the expressions $\operatorname{sh}(l_{\beta j}^{(m)}\gamma)$, $\operatorname{sh}(l_{\beta k}^{(m)}\delta)$, ... vanish are eliminated from the treatment. It has been noted in [8, 11] that these frequency values are not associated with any physical discontinuities in the behaviour of an elastic body and they only require a certain change in the form of writing the general solution.

## 4. ASYMPTOTIC ANALYSIS OF THE SOLVING SYSTEM OF INTEGRAL EQUATIONS

After replacing the initial boundary-value problem by the subsidiary problem, defined by boundary conditions (31), part of the boundary conditions (1.1) remained unsatisfied. They can be considered as a system of integral equations in the unknown subsidiary functions $f_1(\hat{y})$, $\varphi_1(\hat{y})$, ... , $f_8(\hat{x})$. These functions can have singularities at the edge points of their domains of definition. By taking account of these singularities we can separate out and sum the slowly converging parts in the series for the wave characteristics and to successfully select the coordinate functions in asymptotic methods for solving a system of integral equations. The nature of the singularities at points $A$, $B$ and $C$ (Fig. 1) has previously been investigated in [7, 10] and, in this paper, we shall therefore formulate the problem of determining the singularity in the wave field at the internal point $D(\delta, \gamma)$ of the joint of the three domains. For this purpose, we shall assume that the functions (3.1), which are asymptotically significant in the neighbourhood of this point, have singularities of the following form

$$f_i'(\xi) = F_i^D\xi^{\alpha-1}, \quad \varphi_j(\xi) = \Phi_j^D\xi^{\alpha-1}; \quad i = 1, 6; \quad j = 1, 2 \text{ when } \xi \to 0$$

$$f_3'(\xi) = F_3^D(\delta-\xi)^{\alpha-1} \text{ when } \xi \to \delta; \quad f_5'(\xi) = F_5^D(\gamma-\xi)^{\alpha-1} \text{ when } \xi \to \gamma$$

In these formulae, $\alpha$ is a parameter which determines the singularities of the above-mentioned functions at the point $D$, and $F_i^D$, $\Phi_j^D$ ($i = 1, 3, 5, 6; j = 1, 2$) are arbitrary constants.

Determining the asymptotic form of the Fourier coefficients of the functions being considered, we write the boundary conditions, which have not been used in the subsidiary problems and the conditions for matching the regions $G^{(m)}$ in the neighbourhood of point $D$, for the limiting values of the arguments, that is,

$$\begin{array}{ll} \sigma_{11}^{(1)}(\delta,\hat{y}) = \sigma_{11}^{(3)}(0,\hat{y}) \text{ when } \hat{y}\to 0; & \sigma_{22}^{(2)}(\hat{x},\gamma) = \sigma_{22}^{(3)}(\hat{x},0) \text{ when } \hat{x}\to 0 \\ U_2^{(1)}(\delta,\hat{y}) = U_2^{(3)}(0,\hat{y}) \text{ when } \hat{y}\to 0; & U_1^{(2)}(\hat{x},\gamma) = U_1^{(3)}(\hat{x},0) \text{ when } \hat{x}\to 0 \\ \sigma_{11}^{(2)}(0,y) = 0 \text{ when } y\to\gamma; & \sigma_{22}^{(1)}(x,0) = 0 \text{ when } x\to\delta \end{array} \tag{4.1}$$

Redefining the constants and taking account of the fact that there are no discontinuities in the external load at this point, we reduce conditions (4.1) to a system of homogeneous equations, which define the nature of the singularities in the wave-field characteristics at the point $D$,

$$\begin{aligned} &-m_{13}s_\alpha\Phi_1 + r_{21}(1+\alpha d_{11}^{(3)})\Phi_2 - 2(d_{11}^{(1)} + r_{31}d_{11}^{(3)})s_\alpha F_1 - 2d_{11}^{(1)}\alpha F_3 - 2r_{31}d_{11}^{(3)}\alpha F_6 = 0 \\ &r_{12}(1+\alpha d_{11}^{(3)})\Phi_1 - m_{23}s_\alpha\Phi_2 - 2r_{32}d_{11}^{(3)}\alpha F_1 - 2d_{11}^{(2)}\alpha F_5 + 2(d_{11}^{(2)} - r_{32}d_{11}^{(3)})s_\alpha F_6 = 0 \\ &-(A^{(1)} + r_{13}A^{(3)})s_\alpha\Phi_1 + r_{23}d_{11}^{(3)}\alpha\Phi_2 + 2m_{13}s_\alpha F_1 + 2(1-\alpha d_{11}^{(1)})F_3 + 2(1-\alpha d_{11}^{(3)})F_6 = 0 \\ &r_{13}d_{11}^{(3)}\alpha\Phi_1 - (A^{(2)} + r_{23}A^{(3)})s_\alpha\Phi_2 + 2(1-\alpha d_{11}^{(3)})F_1 + 2(1-\alpha d_{11}^{(2)})F_5 + 2m_{23}s_\alpha F_6 = 0 \\ &\left(\frac{1}{d_{11}^{(2)}} + \alpha\right)\Phi_2 - 2s_\alpha F_5 - 2\alpha F_6 = 0, \quad \left(\frac{1}{d_{11}^{(1)}} + \alpha\right)\Phi_1 - 2\alpha F_1 - 2s_\alpha F_3 = 0 \end{aligned} \tag{4.2}$$

where

$$d_{11}^{(m)} = \frac{1}{2(1-\nu^{(m)})}, \quad A^{(m)} = 2 - d_{11}^{(m)}, \quad m_{ij} = \frac{2 - 3(\nu^{(i)} + \nu^{(j)}) + 4\nu^{(i)}\nu^{(j)}}{2(1-\nu^{(i)})(1-\nu^{(j)})}$$

$$\Phi_\beta = -2\Phi_\beta^D\Gamma(\alpha)s_\alpha, \quad F_k = 2F_k^D\Gamma(\alpha)s_\alpha, \quad k = 1,3,5,6; \ s_\alpha = \sin\frac{\pi\alpha}{2}$$

and $\Gamma(\alpha)$ is the gamma-function.

In system (4.2), the number of an equation corresponds to the number of the boundary condition in formulae (4.1).

The parameter $\alpha$, which characterizes a singularity in the wave characteristics at the internal corner point of a composite region, can be determined from the condition for a non-trivial solution of system (4.2) to exist

$$\Delta(\alpha, \mu^{(m)}, \nu^{(m)}) = 0 \tag{4.3}$$

It should be noted that the parameter $\alpha$ is independent of the frequency and the geometrical parameters $\gamma$, $\delta$ and $\eta$ and is solely determined by the values of the shear moduli and Poisson's ratios of the joined regions. This conclusion follows from the form of Eq. (4.3) and is determined by the local nature of the singularity: this is also confirmed by the fact that Eq. (4.3) does not change its form when $\mu^{(1)}$ and $\nu^{(1)}$ are changed to $\mu^{(2)}$ and $\nu^{(2)}$ and back again. This can be proved using elementary transformations of the rows and columns of the determinant of system (4.2)

## 5. NUMERICAL ANALYSIS OF THE SINGULARITY PARAMETER

In the numerical analysis of problems of the type being considered, the main attention is given to investigating the spectrum of resonance frequencies and the maximum dynamic stresses. However, a numerical investigation of Eq. (4.3) with the aim of determining the parameter for the local singularity at the internal corner point of the section is also of interest. It shows that, for certain ratios of the constant of elasticity of the regions $G^{(m)}$ which join at point $D$, Eq. (4.3) has a real root $\alpha$, $0 < \alpha < 1$. It

Table 1

| Material of the regions $G^{(3)}$ and $G^{(2)}$ | Material of the region $G^{(3)}$ | | | | |
|---|---|---|---|---|---|
| | Brass | Tin | Platinum | Lead | Zinc |
| Aluminium | 0.985 | 0.971 | 0.968 | 0.952 | 0.994 |
| Tungsten | 0.965 | 0.976 | 0.962 | 0.901 | 0.990 |
| Magnesium | 0.984 | 0.992 | 0.979 | 0.897 | 0.971 |
| Copper | 0.986 | 0.981 | 0.994 | 0.908 | 1.106 |
| Nickel | 0.993 | 0.951 | 0.997 | 0.878 | 1.018 |
| Silver | 0.971 | 0.987 | 0.975 | 0.922 | 0.993 |
| Steel 20 | 0.991 | 0.942 | 0.995 | 0.902 | 1.009 |

characterizes the emergence of local singularities in the stresses at this point. Since, at point $D$, we have the union of three unlike regions at once, it is not possible in this case to introduce compact parameters analogous to Dunders coefficients [1], which determine the existence of a discontinuity at a singular point of the boundary where two regions join.

The data from calculations of the roots of Eq. (4.3) for various combinations of elastic properties of the materials of the regions $G^{(m)}$ are shown in Table 1. The case when the material of regions $G^{(1)}$ and $G^{(2)}$ are the same, which is most frequently encountered in practice, is considered here.

The question of how the stress singularity parameter depends on the ratio of the stiffnesses of the regions being joined is of practical interest. If the elasticity parameters of the regions $G^{(1)}$ and $G^{(2)}$ are fixed ($\nu^{(1)} = \nu^{(2)} = \nu^{(3)}$, $\mu^{(1)} = \mu^{(2)}$, $r_{32} = r_{31}$) and taken as being equal to those of steel, and only the shear modulus of the corner region $G^{(3)}$ is varied, we arrive at the data in Fig. 2, where the relation $\alpha = \alpha(r_{32})$ is shown.

It follows from an analysis of this relation that a local singularity in the stresses at the internal corner point appears for any $r_{32}$. Analysis of the data in Fig. 2 enables one, already at the design state, to make an optimal choice of the stiffnesses of the materials of the joined structural components with the aim of reducing the stress concentrations at problem points of their sections.

We will now determine the asymptotic form of the singularity parameter $\alpha$ for large values of the shear modulus of the corner region $\mu^{(3)}$. To do this, we will introduce the small dimensionless parameters $\varepsilon_j = \mu^{(j)}/\mu^{(3)} = r_{3j}^{-1}$ $(j = 1, 2)$ and, on expanding the solution of Eq. (4.3) in a power series in this parameter

$$\alpha = \alpha_0 + \varepsilon_1\alpha_{11} + \varepsilon_2\alpha_{12} + \dots \tag{4.4}$$

we can quite easily obtain a sequence of equations for determining $\alpha_0$, $\alpha_{11}$, $\alpha_{12}$, … . For example, the first term in the expansion $\alpha_0$ satisfies the equation

$$\left(\sin^2\frac{\pi\alpha_0}{2} - \alpha_0^2\right)\prod_{i=1}^{2}\left(\alpha_0^2 - (3 - 4\nu^{(i)})\sin^2\frac{\pi\alpha_0}{2} - 4(1 - \nu^{(i)})^2\right) = 0 \tag{4.5}$$

The first factor in Eq. (4.5) is identical to the left-hand side of a well-known equation [1, 7, 8, 10] which determines a singularity in the components of the stress tensor at the vertex of a homogeneous wedge with an aperture angle of 90°. Its roots are independent of the constants of elasticity of the material and, when constructing the asymptotic form of the solution, it is only necessary to take account of the real root $\alpha_0 = 1$ of this equation and the denumerable set of complex roots with a positive real part. Numerical analysis shows that the second and third factors only have roots $\alpha_0 \in (0, 1)$ when $\nu^{(j)} > 0.62$, which does into correspond to the elasticity parameters of real materials. Hence, it can be stated that the local singularity parameter tends to unity at large values of $\alpha_{11}$, $\alpha_{12}$. We also note that the results obtained are only of a qualitative nature since the determination of the following terms of the asymptotic form in expansion (4.4) leads to equations containing the sum of four determinants with elements which depend on $\alpha_0$ and $\alpha_1$. Its numerical solution is far more difficult than the solution of Eq. (4.3).

## 6. NUMERICAL INVESTIGATION OF THE WAVE-FIELD CHARACTERISTICS

The system of integral equations is solved using the conventional method [7, 8] by reducing it to an infinite system of linear algebraic equations in the Fourier coefficients of the subsidiary functions. The

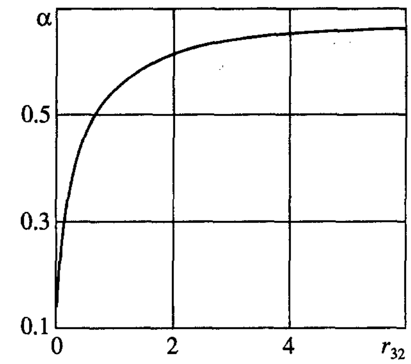


Fig. 2

Table 2

| No of resonance frequencies | Steel | | Steel–Aluminium–Steel | | Steel–Silver–Steel | | Steel–Gold–Steel | |
|---|---|---|---|---|---|---|---|---|
| | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ | $\Omega^{(1)}$ |
| 1 | 0.303 | 0.301 | 0.267 | 0.264 | 0.319 | 0.320 | 0.333 | 0.330 |
| 2 | 0.821 | 0.814 | 0.743 | 0.732 | 0.869 | 0.863 | 0.897 | 0.889 |
| 3 | 1.133 | 1.120 | 1.007 | 0.992 | 1.102 | 1.092 | 1.296 | 1.290 |

known asymptotic form of the behaviour of these coefficients in the case of high numbers enables one to reduce this system to a finite system. We obtain the frequency equation by equating the determinant of the system to zero. The error in satisfying the conditions for the media to be matched served as the criterion for the reliability of the calculations. The error in satisfying the matching conditions with respect to the displacements did not exceed 2–3% of the maximum magnitude of the displacements over the whole frequency band being considered. The accuracy in satisfying the matching conditions with respect to the stresses in the neighbourhood of the singular point $A$, $C$ and $D$ of the boundary of the composite region did not exceed 6–8%.

In order to ensure the reliability of the results obtained, calculations were also carried out by the finite element method using the ANSYS program. The method of rigid body modelling was used in setting up the finite-element model (the geometrical boundaries of the model are described and the program then generates a mesh with nodes and elements and the dimensions and shape of the elements can be monitored). The object is modelled by the six nodal triangular rigid body elements PLANE 2. Modal analysis assists in determining the parameters of the oscillations of the composite region: the characteristic frequencies and the modes of the vibrations are determined using it. The reliability with respect to the finite-element model was monitored by changing the density of the mesh and comparing the results obtained. For practically all geometrical dimensions, it was sufficient to specify no more than 900 nodes.

The values of the first, second and third characteristic frequencies for a homogeneous section of a steel component ($\eta = 0.5$, $\delta = 0.4$, $\gamma = 0.2$) and various combinations of materials in the regions being joined are shown in Table 2 (the material of region $G^{(1)}$ is in the first space, the material of region $G^{(2)}$ is in the second space and the material of region $G^{(3)}$ is in the third space) as obtained using the proposed method $\Omega_i^{(1)}$ and found using the finite element method $\Omega_i^{(1)\prime}$. It follows from the data in Table 2 that the agreement between the results is quite good, but the values of the frequencies $\Omega_i^{(1)\prime}$ obtained were somewhat lower than $\Omega_i^{(1)}$ for practically all combinations of materials considered. As might have been expected, the errors become larger as the number of the characteristic frequency increases.

When investigating the spectrum of the resonance frequencies of the oscillations of the composite regions, it is of particular interest to determine the frequencies at which the occurrence of intense oscillations, localized in the neighbourhood of the interfaces, are characteristic. In the case of the matching of two regions, these questions have previously been investigated in [5–7].

We will now present an analysis of the mean energy after a period $\bar{E}$ [7, 8] which is accumulated in the region $G^{(1-3)} = \{|x - \delta| \leq 0.1\delta_2;\ 0 \leq \hat{y} \leq \gamma_2\}$, that is, in the neighbourhood of the interface $DA$ of

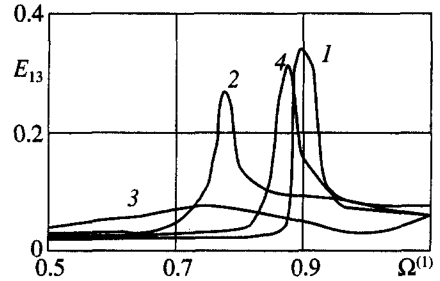


Fig. 3

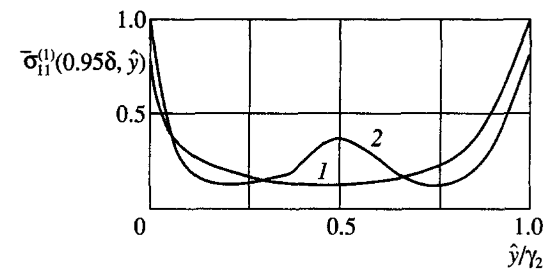


Fig. 4

regions $G^{(1)}$ and $G^{(3)}$ (Fig. 1). The frequency dependence of the ratio $E_{13} = \bar{E}^{(1-3)}/\bar{E}$ of the energy accumulated in the region $G^{(1-3)}$ to the energy accumulated in the whole section is shown in Fig. 3. Calculations were carried out for various combinations of the elasticity parameters of the joined regions when $\eta = 0.7$, $\delta = 0.8$, $\gamma = 0.5$.

Curve I corresponds to the case when the material of the regions $G^{(1)}$ and $G^{(2)}$ is steel and the material of region $G^{(3)}$ is lead (a steel–steel–lead combination). A rather abrupt jump in the energy is observed in the neighbourhood of the frequency $\Omega^{(1)} = 0.897$, which is explained by the occurrence of intense oscillations, localized in the neighbourhood of the interface of the regions. It is natural to refer to such frequencies as boundary resonance frequencies [5]. The magnitude of the maximum of the ratio $\bar{E}^{(1-3)}/\bar{E}$ depends very much on the elasticity properties of the contacting materials. Thus, in the case of the triplet of materials steel–steel–brass, the intensity of the oscillations at the boundary resonance frequency is reduced (curve 2, Fig. 3). At the same time, the boundary resonance frequency is shifted somewhat compared with the preceding case, which is associated with the change in the parameters. In the case of pairs of materials which correspond to values of the singularity parameter $\alpha > 1$, practically no jump is observed in the magnitude of the coefficient for the singularity in the stresses. Curve 3 in Fig. 3 corresponds to the frequency dependence of the energy ratio for the triplet of materials, steel–steel–zinc.

It should be noted that, as might have been expected, a change in the elasticity parameters of region $G^{(2)}$ has only a small effect on the frequency dependence of the energy accumulated in region $G^{(1-3)}$. In the majority of various which have been considered, the maximum energy jump is somewhat reduced compared with the case of identical materials in regions $G^{(1)}$ and $G^{(2)}$. Curve 4 in Fig. 3 shows the above-mentioned dependence for the triplet of materials steel–aluminium–lead.

In the numerical analysis of the dynamic components of the stress tensor, it is necessary to take account not only of the geometric and elasticity parameters of the contacting regions but also the values of the parameters of the local stress singularity at the singular points of the section. It is no less important to take account of the acoustic impedances of the contacting media [6] since, when they have different ratios, a different degree of reflection of the wave field from the interface is observed. The distribution of the normalized stresses (the stresses referred to the maximum stress) $\sigma_{11}^{(1)}$ $(0.95\delta, \hat{y})$, calculated for the first ($\Omega^{(1)} = 0.333$, curve 1) and second ($\Omega^{(1)} = 0.897$, curve 2) resonance frequencies for a constant intensity of the external load, is shown in Fig. 4. The combination of materials is steel–gold–steel and the geometric parameters of the section are equal to $\eta = 0.5$, $\delta = 0.4$, $\gamma = 0.2$.

It follows from the data in Fig. 4 that the maximum shear stresses arise in the neighbourhood of the different singular points of the section: for the first characteristic frequency, at point $A$ and for the second characteristic frequency, at point $D$. This feature is also preserved in the case of other combinations of elastic and geometric parameters of the composite region.

## 7. CONCLUSIONS

The determination of the real roots of Eq. (4.3) enables one to predict the nature of the dynamic concentration of stresses in the unsafe zones of a section of prismatic composite solids. By selecting the elasticity characteristics of the joined regions, which correspond to the maximum values of the local singularity parameter $\alpha$, it is possible to minimize the dynamic stresses at the singular points of the boundary of a section, which correspond to the internal corner points of the joint of unlike materials. The results obtained can be used when designing welded, soldered and glued corner joints operating in a vibration field.

## REFERENCES

1. BOGY, D. B., Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *Trans. ASME. Ser. E. J. Appl. Mech.* 1971, **38**, 171–180.
2. LOBODA, V. V., The contact interaction of an elastic rectangular plate and a strip. *Prikl. Mekh.*, 1989, **25**, 4, 69–76.
3. CHOBANYAN, K. S. and GEVORKYAN, S. Kh., The behaviour of the stress field around a corner point of the line of separation in the problem of the plane deformation of a composite elastic solid. *Izv. Akad. Nauk ArmSSR*, 1971, **24**, 5, 16–24.
4. AKSENTYAN, O. K., Singularities of the stress–strain state of a plate in the neighbourhood of a rib. *Prikl. Mat. Mekh.*, 1967, **31**, 1, 178–186.
5. GETMAN, I. P. and LISITSKII, O. N., Reflection and transmission of sound waves across the interface of two joined elastic half-strips. *Prikl. Mat. Mekh.*, 1988, **52**, 6, 1044–1048.
6. GRICHENKO, V. T. and GORODETSKAYA, N. S., Reflection of Lamb waves from the interface in a composite waveguide. *Prikl. Mekh.*, 1985, **21**, 5, 121–125.
7. VOVK, L. P., Characteristics of the harmonic oscillations of a piecewise-inhomogeneous rectangular domain. *Izv. Vuzov. Sev.-Kavkaz. Regiona. Yestestv. Nauki*, 2002, 4, 9–13.
8. GRICHENKO, V. T. and MELESHKO, V. V., *Harmonic Oscillations and Waves in Elastic Bodies*. Naukova Dumka, Kiev, 1981.
9. ASNIS, A. Ye. and MOSENKIS, Yu. G., *Reduction of the Metal Content in the Welded Joints of Structures*. Tekhnika, Kiev, 1987.
10. VOVK, L. P., Asymptotic investigation of the characteristic oscillations of an inhomogeneous rectangle with an internal aperture. *Izv. Vuzov Sev.-Kavkaz. Regiona. Yestestv. Nauki*, 2001, 1, 29–33.
11. BELOKON', A. V., A method of solving problems in the theory of elasticity for bodies of finite dimensions. *Dokl. Akad. Nauk SSSR*, 1977, **233**, 1, 56–59.

*Translated by* E.L.S.